

# The Marstrand Theorem in Nonpositive Curvature

Sergio Augusto Rom  a Ibarra\*

February 21, 2014

## Abstract

In a paper from 1954, Marstrand proved that if  $K \subset \mathbb{R}^2$  with Hausdorff dimension greater than 1, then its one-dimensional projection has positive Lebesgue measure for almost-all directions. In this article, we show that if  $M$  is a simply connected surface with non-positive curvature, then Marstrand’s theorem is still valid.

## 1 Introduction

Consider  $\mathbb{R}^2$  as a metric space with a metric  $d$ . If  $U$  is a subset of  $\mathbb{R}^2$ , the diameter of  $U$  is  $|U| = \sup\{d(x, y) : x, y \in U\}$  and, if  $\mathcal{U}$  is a family of subsets of  $\mathbb{R}^2$ , the diameter of  $\mathcal{U}$  is defined by

$$\|\mathcal{U}\| = \sup_{U \in \mathcal{U}} |U|.$$

Given  $s > 0$ , the Hausdorff  $s$ -measure of a subset  $K$  of  $\mathbb{R}^2$  is

$$m_s(K) = \lim_{\epsilon \rightarrow 0} \left( \inf_{\substack{\mathcal{U} \text{ covers } K \\ \|\mathcal{U}\| < \epsilon}} \sum_{U \in \mathcal{U}} |U|^s \right).$$

In particular, when  $d$  is the Euclidean metric and  $s = 1$ , then  $m = m_1$  is the Lebesgue measure. It is not difficult to show that there exists a unique  $s_0 \geq 0$  for which  $m_s(K) = +\infty$  if  $s < s_0$  and  $m_s(K) = 0$  if  $s > s_0$ . We define the Hausdorff dimension of  $K$  as  $HD(K) = s_0$ . Also, for each  $\theta \in \mathbb{R}$ , let  $v_\theta = (\cos \theta, \sin \theta)$ ,  $L_\theta$  the line in  $\mathbb{R}^2$  through of the origin containing  $v_\theta$  and  $\pi_\theta : \mathbb{R}^2 \rightarrow L_\theta$  the orthogonal projection.

In 1954, J. M. Marstrand [Mar54] proved the following result on the fractal dimension of plane sets.

**Theorem[Marstrand]:** *If  $K \subset \mathbb{R}^2$  such that  $HD(K) > 1$ , then  $m(\pi_\theta(K)) > 0$  for  $m$ -almost every  $\theta \in \mathbb{R}$ .*

The proof is based on a qualitative characterization of the “bad” angles  $\theta$  for which the result is not true.

---

\*Partially supported by CNPq, Capes and the Palis Balzan Prize.

Many generalizations and simpler proofs have appeared since. One of them came in 1968 by R. Kaufman, who gave a very short proof of Marstrand's Theorem using methods of potential theory. See [Kau68] for his original proof and [PT93], [Fal85] for further discussion. Another recent proof of the theorem (2011), which uses combinatorial techniques is found in [LM11].

In this article, we consider  $M$  a simply connected surface with a Riemannian metric of non-positive curvature, and using the potential theory techniques of Kaufman [Kau68], we show the following more general version of the Marstrand's Theorem.

**The Geometric Marstrand Theorem:** *Let  $M$  be a Hadamard surface, let  $K \subset M$  and  $p \in M$ , such that  $HD(K) > 1$ , then for almost every line  $l$  coming from  $p$ , we have  $\pi_l(K)$  has positive Lebesgue measure, where  $\pi_l$  is the orthogonal projection on  $l$ .*

Then using the Hadamard's theorem (cf. [PadC08]), the theorem above can be stated as follows:

**Main Theorem:** *Let  $\mathbb{R}^2$  be endowed with a metric  $g$  of non-positive curvature, and  $K \subset \mathbb{R}^2$  with  $HD(K) > 1$ . Then for almost every  $\theta \in (-\pi/2, \pi/2)$ , we have that  $m(\pi_\theta(K)) > 0$ , where  $\pi_\theta$  is the orthogonal projection with the metric  $g$  on the line  $l_\theta$ , of initial velocity  $v_\theta = (\cos \theta, \sin \theta) \in T_p \mathbb{R}^2$ .*

## 2 Preliminaries

Let  $M$  be a Riemannian manifold with metric  $\langle \cdot, \cdot \rangle$ , a line in  $M$  is a geodesic defined for all parameter values and minimizing distance between any of its points, that is,  $\gamma : \mathbb{R} \rightarrow M$  is a isometry. If  $M$  is a manifold of dimension  $n$ , simply connected and non-positive curvature, then the space of lines leaving of a point  $p$  can be seen as a sphere of dimension  $n - 1$ . So, in the case of surfaces the set of lines agrees with  $S^1$  in the space tangent  $T_p M$  of the point  $p$ . Therefore, in each point on the surface the set of lines can be oriented and parameterized by  $(-\frac{\pi}{2}, \frac{\pi}{2}]$  and endowed of Lebesgue measure. Thus, using the previous identification, we can talk about almost every line through a point of  $M$  (cf. [BH99]). In the conditions above, Hadamard's theorem states that  $M$  is diffeomorphic to  $\mathbb{R}^n$ , (cf. [PadC08]).

Moreover, given a geodesic triangle  $\triangle ABC$  with sides  $\vec{BC}$  and  $\vec{AC}$  denote by  $\angle A$  the angle between geodesic segments  $\vec{AB}$  and  $\vec{AC}$ , then *the law of cosines* says

$$|\vec{BC}|^2 \geq |\vec{AB}|^2 + |\vec{AC}|^2 - 2|\vec{AB}||\vec{AC}| \cos \angle A,$$

where  $|\vec{ij}|$  is the distance between the points  $i, j$  for  $i, j \in \{A, B, C\}$ .

**Gauss's Lemma:** Let  $p \in M$  and let  $v, w \in B_\epsilon(0) \in T_p M \approx T_p M$  and  $M \ni q = \exp_p v$ . Then,

$$\langle d(\exp_p)_v v, d(\exp_p)_v w \rangle_q = \langle v, w \rangle_p.$$

## 2.1 Projections

Let  $M$  be a manifold simply connected and of non-positive curvature. Let  $C$  be a complete convex set in  $M$ . The *orthogonal projection* (or simply ‘*projection*’) is the name given to the map  $\pi: M \rightarrow C$  constructed in the following proposition: (cf. [BH99, pp 176]).

**Proposition 1.** *The projection  $\pi$  satisfies the following properties:*

1. *For any  $x \in M$  there is a unique point  $\pi(x) \in C$  such that  $d(x, \pi(x)) = d(x, C) = \inf_{y \in C} d(x, y)$ .*
2. *If  $x_0$  is in the geodesic segment  $[x, \pi(x)]$ , then  $\pi(x_0) = \pi(x)$ .*
3. *Given  $x \notin C$ ,  $y \in C$  and  $y \neq \pi(x)$ , then  $\angle_{\pi(x)}(x, y) \geq \frac{\pi}{2}$ .*
4.  *$x \mapsto \pi(x)$  is a retraction on  $C$ .*

**Corollary 1.** *Let  $M, C$  be as above and define  $d_C(x) := d(x, C)$ , then*

1.  *$d_C$  is a convex function, that is, if  $\alpha(t)$  is a geodesic parametrized proportionally to arc length, then*

$$d_C(\alpha(t)) \leq (1 - t)d_C(\alpha(0)) + td_C(\alpha(1)) \text{ for } t \in [0, 1].$$

2. *For all  $x, y \in M$ , we have  $|d_C(x) - d_C(y)| \leq d(x, y)$ .*
3. *The restriction of  $d_C$  to the sphere of center  $x$  and radius  $r \leq d_C(x)$  reaches the infimum in a unique point  $y$  with*

$$d_C(x) = d_C(y) + r.$$

Here we consider  $\mathbb{R}^2$  with a Riemannian metric  $g$ , such that the curvature  $K_{\mathbb{R}^2}$  is non-positive, i.e.,  $K_{\mathbb{R}^2} \leq 0$ . Recall that a line  $\gamma$  in  $\mathbb{R}^2$  is a geodesic defined for all parameter values and minimizing distance between any of its points, that is,  $\gamma: \mathbb{R} \rightarrow \mathbb{R}^2$  and  $d(\gamma(t), \gamma(s)) = |t - s|$ , where  $d$  is the distance induced by the Riemannian metric  $g$ , in other words, a parametrization of  $\gamma$  is a isometry. Then, given  $x \in \mathbb{R}^2$  there is a unique  $\gamma(t_x)$  such that  $\pi_\gamma(x) = \gamma(t_x)$ , thus without loss of generality we may call  $\pi_\gamma(x) = t_x$ .

Fix  $p \in \mathbb{R}^2$  and let  $\{e_1, e_2\}$  be a positive orthogonal basis of  $T_p\mathbb{R}^2$ , i.e., the basis  $\{e_1, e_2\}$  has the induced orientation of  $\mathbb{R}^2$ . Then, call  $v_t = (\cos t, \sin t)$  in coordinates the unit vector  $(\cos t)e_1 + (\sin t)e_2 \in T_p\mathbb{R}^2$ . Denote by  $l_t$  the line through  $p$  with velocity  $v_t$ , given by  $l_t(s) = \exp_p s v_t$  and by  $\pi_t$  the projection on  $l_t$ . Then, given  $\theta \in [0, 2\pi)$ , we can define  $\pi: [0, 2\pi) \times T_p\mathbb{R}^2 \rightarrow \mathbb{R}$  by the unique parameter  $s$  such that  $\pi_\theta(\exp_p w) = \exp_p s v_\theta$  i.e.,  $\pi(\theta, w) := \pi_\theta(w)$  and

$$\pi_\theta(\exp_p w) = \exp_p \pi(\theta, w) v_\theta.$$

## 3 Behavior of the Projection $\pi$

In this section we will prove some lemmas that will help to understand the projection  $\pi$ .

### 3.1 Differentiability of $\pi$ in $\theta$ and $w$

**Lemma 1.** *The projection  $\pi$  is differentiable in  $\theta$  and  $w$ .*

**Proof.** Fix  $w$  and call  $q = \exp_p w$ . Let  $\alpha_v(t) \subset T_q \mathbb{R}^2$  such that  $\exp_q \alpha_v(t) = \gamma_v(t)$ , where  $\gamma_v$  is the line such that  $\gamma_v(0) = p$  and  $\gamma'_v(0) = v$ , then, for all  $v \in S^1$ , there is a unique  $t_v$  such that  $d(q, \gamma_v(\mathbb{R})) = d(q, \gamma_v(t_v))$  and satisfies

$$\langle d(\exp_q)_{\alpha_v(t_v)}(\alpha'_v(t_v)), d(\exp_q)_{\alpha_v(t_v)}(\alpha_v(t_v)) \rangle = \langle \gamma'_v(t_v), d(\exp_q)_{\alpha_v(t_v)}(\alpha_v(t_v)) \rangle = 0.$$

By Gauss Lemma, we have

$$\frac{1}{2} \frac{\partial}{\partial t} \|\alpha_v(t)\|^2(t_v) = \langle \alpha'_v(t_v), \alpha_v(t_v) \rangle = 0.$$

We define the real function

$$\begin{aligned} \eta : S^1 \times \mathbb{R} &\longrightarrow \mathbb{R} \\ \eta(v, t) &= \frac{1}{2} \frac{\partial}{\partial t} \|\alpha_v(t)\|^2, \end{aligned}$$

this function is  $C^\infty$  and satisfies  $\eta(v, t_v) = 0$ , also  $\frac{\partial}{\partial t} \eta(v, t) = \frac{1}{2} \frac{\partial^2}{\partial t^2} \|\alpha_v(t)\|^2$ .

Put  $g(t) = \|\alpha_{v_0}(t)\|^2$ , then  $\frac{\partial}{\partial t} \eta(v_0, t_0) = \frac{1}{2} g''(t_0)$ . Also,  $g(t) = d(q, \gamma_{v_0}(t))^2$  is differentiable and has a global minimum at  $t_{v_0}$ , as  $K_{\mathbb{R}^2} \leq 0$ ,  $g$  is convex. In fact, for  $s \in [0, 1]$

$$\begin{aligned} g(sx + (1-s)y) &= d(q, \gamma_{v_0}(sx + (1-s)y))^2 \leq (sd(q, \gamma_{v_0}(x)) + (1-s)d(q, \gamma_{v_0}(y)))^2 \\ &\leq sd(q, \gamma_{v_0}(x))^2 + (1-s)d(q, \gamma_{v_0}(y))^2 = sg(x) + (1-s)g(y) \end{aligned}$$

by the law of cosines and using the fact  $\angle_{\pi_{\gamma_{v_0}(t_0)}}(q, \gamma_{v_0}(t)) = \frac{\pi}{2}$  at the point of projection

$$d(q, \gamma_{v_0}(t_{v_0}))^2 + d(\gamma_{v_0}(t_{v_0}), \gamma_{v_0}(t))^2 \leq d(q, \gamma_{v_0}(t))^2$$

equivalently

$$g(t_{v_0}) + (t - t_{v_0})^2 \leq g(t).$$

Therefore, as  $g'(t_{v_0}) = 0$ , then  $g''(t_{v_0}) > 0$ . This implies that  $\frac{\partial \eta}{\partial t}(v_0, t_0) \neq 0$  and by Theorem of Implicit Functions, there is an open  $U$  containing  $(v_0, t_{v_0})$ , a open  $V \subset S^1$  containing  $v_0$  and  $\xi : V \longrightarrow \mathbb{R}$ , a class function  $C^\infty$  with  $\xi(v_0) = t_{v_0}$  such that

$$\{(v, t) \in U : \eta(v, t) = 0\} \iff \{v \in V : t = \xi(v)\}.$$

Since by construction  $\eta(v, \xi(v)) = 0$  implies  $\pi(v, q) = \xi(v)$ , and therefore  $\pi(v, q)$  is differentiable in  $v$ , in fact it is  $C^\infty$ . The above shows that  $\pi$  is differentiable in  $\theta$ .

Analogously, is proven that  $\pi$  is differentiable in  $w$ .

□

Let  $w \in T_p \mathbb{R}^2 \setminus \{0\}$  and put  $\theta_w^\perp \in [0, 2\pi)$  such that  $w$  and  $v_{\theta_w^\perp}$  are orthogonal, that is  $\langle w, v_{\theta_w^\perp} \rangle = 0$ , where the  $\langle \cdot, \cdot \rangle$  is the inner product in  $T_p \mathbb{R}^2$  and the set  $\{w, v_{\theta_w^\perp}\}$  is a positive basis of  $T_p \mathbb{R}^2$ .

**Lemma 2.** *The projection  $\pi$  satisfies,*

$$\frac{\partial \pi}{\partial \theta}(\theta_w^\perp, w) = -\|w\|.$$

Moreover, there exists  $\epsilon > 0$  such that, for all  $w$

$$-\|w\| \leq \frac{\partial \pi}{\partial \theta}(\theta, w) \leq -\frac{1}{2}\|w\| \quad \text{and} \quad \left| \frac{\partial^2 \pi}{\partial^2 \theta}(\theta, w) \right| \leq \|w\|,$$

whenever  $|\theta - \theta_w^\perp| < \epsilon$ .

Before proving Lemma 2 we will seek to understand the function  $\pi(\theta, w)$ .

Let  $\pi_{l_\theta}$  be the orthogonal projection on the line  $l_\theta$  generated by the vector  $v_\theta$  in  $T_p \mathbb{R}^2$ , in this case,  $\pi_{l_\theta}(w) = \|w\| \cos(\arg(w) - \theta)$ , where  $\arg(w)$  is the argument of  $w$  with relation to  $e_1$  and the positivity of basis  $\{e_1, e_2\}$ .

Now using the law of cosines

$$d(p, \pi_\theta(\exp_p w))^2 + d(\exp_p w, \pi_\theta(\exp_p w))^2 \leq \|w\|^2 = \pi_{l_\theta}(w)^2 + d(\pi_{l_\theta}(w)v_\theta, w)^2,$$

Since,  $K \leq 0$ , then

$$d(\exp_p w, \pi_\theta(\exp_p w)) = d(\exp_p w, \exp_p \pi_\theta(w)v_\theta) \geq d(w, \pi_\theta(w)v_\theta) \geq d(w, \pi_{l_\theta}(w)v_\theta).$$

Joining the previous expressions we obtain

$$d(p, \pi_\theta(\exp_p w))^2 \leq \pi_{l_\theta}(w)^2 \iff \pi_\theta(w)^2 \leq \pi_{l_\theta}(w)^2.$$

Thus, since  $\pi_\theta(w)$  has the same sign as  $\pi_{l_\theta}(w)$ , then

$$\pi_\theta(w) \geq 0 \implies \pi_\theta(w) \leq \pi_{l_\theta}(w) = \|w\| \cos(\arg(w) - \theta); \quad (1)$$

$$\pi_\theta(w) \leq 0 \implies \pi_\theta(w) \geq \pi_{l_\theta}(w) = \|w\| \cos(\arg(w) - \theta). \quad (2)$$

**Proof of Lemma 2.**

As  $\langle w, \theta_w^\perp \rangle = 0$ , then  $\arg(w) - \theta_w^\perp = -\pi/2$ , thus  $\pi(\theta_w^\perp, w) = 0 = \|w\| \cos(-\pi/2)$ . Moreover, as  $\pi(\theta_w^\perp - h, w) \geq 0$  and  $\pi(\theta_w^\perp + h, w) \leq 0$  for  $h > 0$  small, then

$$\frac{\pi(\theta_w^\perp - h, w)}{h} \leq \frac{\|w\| \cos(\arg(w) - (\theta_w^\perp - h))}{h}$$

and

$$\frac{\pi(\theta_w^\perp + h, w)}{h} \geq \frac{\|w\| \cos(\arg(w) - (\theta_w^\perp + h))}{h}.$$

If  $h \rightarrow 0$  in the two previous inequalities we have

$$\frac{\partial \pi}{\partial \theta}(\theta_w^\perp, w) \leq -\|w\| \sin(\arg(w) - \theta_w^\perp) = -\|w\|$$

and

$$\frac{\partial \pi}{\partial \theta}(\theta_w^\perp, w) \geq -\|w\| \sin(\arg(w) - \theta_w^\perp) = -\|w\|.$$

Therefore,

$$\frac{\partial \pi}{\partial \theta}(\theta_w^\perp, w) = -\|w\|. \quad (3)$$

Moreover, for  $h > 0$  small and by the equation (2), we have

$$\begin{aligned} \pi(\theta_w^\perp + h, w) &= \frac{\partial \pi}{\partial \theta}(\theta_w^\perp, w)h + \frac{1}{2} \frac{\partial^2 \pi}{\partial^2 \theta}(\theta_w^\perp, w)h^2 + r(h) \\ &\geq \|w\| \cos(\arg(w) - (\theta_w^\perp + h)) \\ &= \|w\| \left( \frac{\partial}{\partial \theta} \cos(\theta - \arg(w))|_{\theta_w^\perp} h + \frac{1}{2} \frac{\partial^2}{\partial^2 \theta} \cos(\theta - \arg(w))|_{\theta_w^\perp} h^2 + R(h) \right). \end{aligned}$$

The above inequality and equation (3) implies that

$$\frac{\partial^2 \pi}{\partial^2 \theta}(\theta_w^\perp, w)h^2 + r(h) \geq \|w\| \left( \frac{\partial^2}{\partial^2 \theta} \cos(\theta - \arg(w))|_{\theta_w^\perp} h^2 + R(h) \right).$$

Since  $\frac{\partial^2}{\partial^2 \theta} \cos(\theta - \arg(w))|_{\theta_w^\perp} = 0$ , then  $\frac{\partial^2 \pi}{\partial^2 \theta}(\theta_w^\perp, w) \geq 0$ . Analogously, using  $\pi(\theta_w^\perp - h, w)$  and equation (1) we have that  $\frac{\partial^2 \pi}{\partial^2 \theta}(\theta_w^\perp, w) \leq 0$ . So,

$$\frac{\partial^2 \pi}{\partial^2 \theta}(\theta_w^\perp, w) = 0. \quad (4)$$

Using Taylor's expansion of third order for  $\pi(\theta_w^\perp + h, w)$  and  $h > 0$ , the equations (2), (4), and the fact that  $\frac{\partial^3}{\partial^3 \theta} \cos(\theta - \arg(w))|_{\theta_w^\perp} = 1$ , implies that

$$\frac{\partial^3 \pi}{\partial^3 \theta}(\theta_w^\perp, w) \frac{h^3}{6} + r_3(h) \geq \frac{h^3}{6} + R_3(h).$$

Thus,

$$\frac{\partial^3 \pi}{\partial^3 \theta}(\theta_w^\perp, w) \geq 1. \quad (5)$$

Equations (4) and (5) implies that, for any  $w \in T_p \mathbb{R}^2$ , the function  $\frac{\partial \pi}{\partial \theta}(\cdot, w)$  has a minimum in  $\theta = \theta_w^\perp$ , therefore there is  $\epsilon_1 > 0$  such that

$$-\|w\| \leq \frac{\partial \pi}{\partial \theta}(\theta, w) \quad \text{for all } |\theta - \theta_w^\perp| < \epsilon_1. \quad (6)$$

The lemma will be proved if we show the following statements:

1. There is  $\delta_1 > 0$ , such that for all  $\|w\| \geq 1$ ,

$$\frac{\partial \pi}{\partial \theta}(\theta, w) \leq -\frac{1}{2} \|w\|, \quad \text{whenever } |\theta - \theta_w^\perp| < \delta_1.$$

In fact: Let  $1/2 > \beta > 0$ , then by continuity of  $\frac{\partial \pi}{\partial \theta}$ , there is  $\delta_1$  such that

$$\text{if } |\theta - \theta_w^\perp| < \delta_1, \quad \text{then } \frac{\partial \pi}{\partial \theta}(\theta, w) - \frac{\partial \pi}{\partial \theta}(\theta_w^\perp, w) < \beta.$$

Thus,  $\frac{\partial \pi}{\partial \theta}(\theta, w) < \beta - \|w\| < -\frac{1}{2} \|w\|$  for any  $\|w\| \geq 1$ .

2. There is  $\epsilon_2 > 0$ , such that for all  $\|w\| = 1$  and  $t \in [0, 1]$

$$\frac{\partial \pi}{\partial \theta}(\theta, tw) \leq -\frac{1}{2}t, \quad \text{whenever } |\theta - \theta_w^\perp| < \epsilon_2.$$

In fact: Suppose by contradiction that for all  $n \in \mathbb{N}$ , there are  $w_n, t_n, \theta_n$ ,  $\|w_n\| = 1$  such that  $|\theta_{w_n}^\perp - \theta_n| < \frac{1}{n}$  and  $\frac{\partial \pi}{\partial \theta}(\theta_n, t_n w_n) > -\frac{1}{2}t_n$ . Without loss of generality, we can assume that  $w_n \rightarrow w$ ,  $\theta_n \rightarrow \theta_w^\perp$  and  $t_n \rightarrow t$ . If  $t \neq 0$ , the above implies a contradiction with (3). Thus, suppose that  $t = 0$ , then consider the  $C^1$ -function  $H(\theta, t, w) = \frac{\partial \pi}{\partial \theta}(\theta, tw)$ , then  $\frac{\partial H}{\partial t}(\theta_w^\perp, 0, w) = -\|w\| = -1$ . Since  $H$  is  $C^1$ , then

$$\lim_{n \rightarrow \infty} \frac{H(\theta_n, t_n, w_n)}{t_n} = \lim_{t \rightarrow 0} \frac{H(\theta, t, w)}{t} = -1 < -1/2 \leq \lim_{n \rightarrow \infty} \frac{H(\theta_n, t_n, w_n)}{t_n}.$$

Which is absurd, so the assertion 2 is proved.

Take  $\epsilon = \min\{\epsilon_1, \epsilon_2, \delta_1\}$ , then by the equation (6) and the statements 1 and 2 we have the second part of Lemma 2. The third part is analogous, just consider that  $\frac{\partial^2 \pi}{\partial^2 \theta}(\theta_w^\perp, w) = 0$ . So we conclude the proof of Lemma.  $\square$

**Lemma 3.** Let  $w \neq 0$  and  $\theta \neq \theta_w^\perp$ , then  $\lim_{t \rightarrow 0^+} \frac{\pi_\theta(tw)}{t} \neq 0$ .

**proof.** Suppose that  $\lim_{t \rightarrow 0^+} \frac{\pi_\theta(tw)}{t} = 0$ , put  $w(t) = \exp_p tw$ , let  $v(t) \in T_{w(t)}\mathbb{R}^2$  the unit vector such that  $\exp_{w(t)} s(t)v(t) = \pi_\theta(\exp_p tw)$  for some  $s(t) \geq 0$ . Let  $J(t) \in T_{w(t)}\mathbb{R}^2$  such that  $\exp_{w(t)} J(t) = p$ , that is,  $J(t) = -d(\exp_p)_{tw} w$ . Then, putting  $\alpha(t)$  the oriented angle between  $v(t)$  and  $J(t)$  (cf. Figure 1).

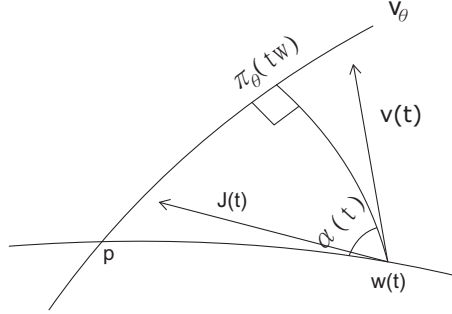


Figure 1: Convergence geodesics

By the law of cosines and using that  $d(p, w(t)) = \|J(t)\| = t\|w\|$  for  $t > 0$ , and  $\pi_\theta(tw) = d(p, \pi_\theta(w(t)))$ , we obtain

$$\pi_\theta(tw)^2 \geq \|J(t)\|^2 + d(w(t), \pi_\theta(w(t)))^2 - 2\|J(t)\|d(w(t), \pi_\theta(w(t)))\cos\alpha(t).$$

Put  $\lim_{t \rightarrow 0^+} \frac{d(w(t), \pi_\theta(w(t)))}{t} = B$ , then dividing by  $t^2$  and when  $t \rightarrow 0$  we have

$$\begin{aligned} 0 &= \left( \lim_{t \rightarrow 0^+} \frac{\pi_\theta(tw)}{t} \right)^2 \geq \|w\|^2 + B^2 - 2\|w\|B \lim_{t \rightarrow 0^+} \cos\alpha(t) \\ &\geq \|w\|^2 + B^2 - 2\|w\|B = (\|w\| - B)^2 \geq 0. \end{aligned}$$

Thus, we conclude that  $B = \|w\|$  and  $\lim_{t \rightarrow 0^+} \cos \alpha(t) = 1$ . Therefore,  $\lim_{t \rightarrow 0^+} \alpha(t) = 0$ , this implies the following geodesic convergence

$$\exp_{w(t)} s v(t) \xrightarrow{t \rightarrow 0^+} \exp_p s \frac{-w}{\|w\|},$$

given that  $w(t) \rightarrow p$  and  $v(t) \rightarrow -\frac{w}{\|w\|}$  when  $t \rightarrow 0^+$ .

Moreover, by definition of  $s(t)$ , we have that

$$\left\langle d(\exp_{w(t)})_{s(t)v(t)} v(t), d(\exp_p)_{\pi_\theta(tw)v_\theta} v_\theta \right\rangle_{\pi_\theta(tw)v_\theta} = 0,$$

using the fact that  $d(\exp_p)_0 = I$ , where  $I$  is the identity of  $T_p \mathbb{R}^2$ , then when  $t \rightarrow 0^+$  and we conclude that  $\left\langle -\frac{w}{\|w\|}, v_\theta \right\rangle = 0$  and this is a contradiction as  $\theta \neq \theta_w^\perp$ .  $\square$

Now we subdivide  $T_p \mathbb{R}^2$  in three regions: Consider  $\epsilon$  given by the Lemma 2, then

$$\begin{aligned} R_1 &= \left\{ w \in T_p \mathbb{R}^2 : \text{the angle } \angle(w, e_1) \leq \frac{\pi}{2} - \frac{3}{2}\epsilon \text{ and } \angle(w, e_1) \geq \frac{3\pi}{2} + \frac{3}{2}\epsilon \right\}; \\ R_2 &= \left\{ w \in T_p \mathbb{R}^2 : \text{the angle } \frac{\pi}{4} + \frac{3}{2}\epsilon \leq \angle(w, e_1) \leq \frac{5\pi}{4} - \frac{3}{2}\epsilon \right\}; \\ R_3 &= \left\{ w \in T_p \mathbb{R}^2 : \text{the angle } \frac{3\pi}{4} + \frac{3}{2}\epsilon \leq \angle(w, e_1) \leq \frac{7\pi}{4} - \frac{3}{2}\epsilon \right\}. \end{aligned}$$

For  $w \in T_p \mathbb{R}^2$ , putting  $a_w^\perp = \theta_w^\perp - \frac{\epsilon}{2}$  and  $\tilde{a}_w^\perp = \theta_w^\perp + \frac{\epsilon}{2}$ , where  $\epsilon$  is given in Lemma 2.

**Lemma 4.** *For the function  $\pi_\theta(w)$  we have that*

1. *There is  $C_1 > 0$  such that for all  $w \in R_1$ ,*

- (a) *If  $\|w\| \leq 1$ , then  $\pi_\theta(w) \geq C_1 \|w\|$  for  $\theta \in [0, a_w^\perp] \cup [\tilde{a}_w^\perp, \pi]$ .*
- (b) *If  $\|w\| \geq 1$ , then  $\pi_\theta(w) \geq C_1$  for  $\theta \in [0, a_w^\perp] \cup [\tilde{a}_w^\perp, \pi]$ .*

2. *There is  $C_2 > 0$  such that for all  $w \in R_2$ ,*

- (a) *If  $\|w\| \leq 1$ , then  $\pi_\theta(w) \geq C_2 \|w\|$  for  $\theta \in [\frac{3}{4}\pi, a_w^\perp] \cup [\tilde{a}_w^\perp, \frac{7}{4}\pi]$ .*
- (b) *If  $\|w\| \geq 1$ , then  $\pi_\theta(w) \geq C_2$  for  $\theta \in [\frac{3}{4}\pi, a_w^\perp] \cup [\tilde{a}_w^\perp, \frac{7}{4}\pi]$ .*

3. *There is  $C_3 > 0$  such that for all  $w \in R_3$ ,*

- (a) *If  $\|w\| \leq 1$ , then  $\pi_\theta(w) \geq C_3 \|w\|$  for  $\theta \in [\frac{5}{4}\pi, a_w^\perp] \cup [\tilde{a}_w^\perp, \frac{9}{4}\pi]$ .*
- (b) *If  $\|w\| \geq 1$ , then  $\pi_\theta(w) \geq C_3$  for  $\theta \in [\frac{5}{4}\pi, a_w^\perp] \cup [\tilde{a}_w^\perp, \frac{9}{4}\pi]$ .*

We prove the part 1, the parts 2 and 3 are analogous.



**proof.** It suffices to prove that there is  $C_1 > 0$  such that for all  $w \in R_1$  with  $\|w\| = 1$  and all  $t \in [0, 1]$  we have

$$\pi_\theta(tw) \geq C_1 t \quad \text{for } \theta \in [0, a_w^\perp] \cup [\tilde{a}_w^\perp, \pi]. \quad (7)$$

In fact: By contradiction, suppose that for all  $n \in \mathbb{N}$  there is  $w_n$  with  $\|w_n\| = 1$ ,  $t_n \in [0, 1]$  and  $\theta_n \in [0, a_{w_n}^\perp] \cup [\tilde{a}_{w_n}^\perp, \pi]$  such that  $\pi_{\theta_n}(t_n w_n) < \frac{1}{n} t_n$ . We can assume that  $w_n \rightarrow w$ ,  $\theta_n \rightarrow \theta \in [0, a_w^\perp] \cup [\tilde{a}_w^\perp, \pi]$  and  $t_n \rightarrow t$  when  $n \rightarrow \infty$ . If  $t \neq 0$ , then since for  $w \in R_1$  and  $\theta \in [0, a_w^\perp] \cup [\tilde{a}_w^\perp, \pi]$ ,  $\pi_\theta(tw) \geq 0$ , we have  $0 \leq \pi_\theta(tw) \leq 0$ , so  $\theta = \theta_w^\perp$  and this is a contradiction, because  $\theta \in [0, a_w^\perp] \cup [\tilde{a}_w^\perp, \pi]$  and  $\epsilon$  is fixed.

If  $t = 0$ , consider the  $C^1$ -function  $F(\theta, t, w) = \pi_\theta(tw)$ , then

$$0 = \lim_{n \rightarrow \infty} \frac{F(\theta_n, t_n, w_n)}{t_n} = \lim_{t \rightarrow 0} \frac{F(\theta, t, w)}{t},$$

by Lemma 3 we know that  $\lim_{t \rightarrow 0} \frac{F(\theta, t, w)}{t} \neq 0$ , and this is a contradiction with the above, so the affirmation is proved.

Now, since  $\theta_w^\perp = \theta_{tw}^\perp$  for  $t > 0$  we have

(a) If  $\|w\| \leq 1$ , then by (7),  $\pi_\theta(w) = \pi_\theta(\|w\| \frac{w}{\|w\|}) \geq C_1 \|w\|$  for  $\theta \in [0, a_w^\perp] \cup [\tilde{a}_w^\perp, \pi]$ .

(b) Since  $\pi_\theta(w) \geq \pi_\theta(\frac{w}{\|w\|})$  for  $\|w\| \geq 1$ , then the equation (7) and implies the result.  $\square$

### 3.2 The Bessel Function Associated to $\pi_\theta(w)$

For  $w \in T_p \mathbb{R}^2$  consider the Bessel function

$$\tilde{J}_w(z) = \int_0^{2\pi} \cos(z\pi_\theta(w)) d\theta.$$

Observe that we can consider  $\pi_\theta(w)$  as a periodic function in  $\theta$  of period  $2\pi$ . Moreover,  $\tilde{J}_w(z)$  has the following properties:

1.  $\tilde{J}_w(z) = \tilde{J}_w(-z)$ ;
2.  $\tilde{J}_w(z) = \int_0^{2\pi} \cos(z\pi_\theta(w)) d\theta = \int_t^{2\pi+t} \cos(z\pi_\theta(w)) d\theta$  for any  $t \in \mathbb{R}$ .
3. As  $\pi_{\theta+\pi}(\exp_p(w)) = -\pi_\theta(\exp_p(w))$ , then

$$\begin{aligned} \int_t^{\pi+t} \cos(z\pi_\theta w) d\theta &= \int_{\pi+t}^{2\pi+t} \cos(z\pi_{\theta-\pi} w) d\theta = \int_{\pi+t}^{2\pi+t} \cos(-z\pi_\theta w) d\theta \\ &= \int_{\pi+t}^{2\pi+t} \cos(z\pi_\theta(w)) d\theta, \end{aligned}$$

Thus,

$$\tilde{J}_w(z) = 2 \int_t^{\pi+t} \cos(z\pi_\theta(w)) d\theta := 2J_w^t(z). \quad (8)$$

**Remark 1.** To fix ideas we consider

$$\begin{aligned} t &= 0 \quad \text{for } w \in R_1; \\ t &= \frac{3}{4}\pi \quad \text{for } w \in R_2; \\ t &= \frac{5}{4}\pi \quad \text{for } w \in R_3. \end{aligned}$$

**Proposition 2.** For any  $w \in T_p \mathbb{R}^2$  we have that  $\int_{-\infty}^{\infty} \tilde{J}_w(z) dz < \infty$ .

**proof.** We divide the proof in three parts.

1. If  $w \in R_1$ , in this case, by Remark 1 and equation (8) it suffices to prove the Lemma for  $J_w^0(z) := J_w(z)$ .
2. If  $w \in R_2$ , in this case, by Remark 1 and equation (8) it suffices to prove the Lemma for  $J_w^{3\pi/4}(z)$ .
3. If  $w \in R_3$ , in this case, by Remark 1 and equation (8) it suffices to prove the Lemma for  $J_w^{5\pi/4}(z)$ .

We will prove 1, the proof of 2 and 3 are analogous. In fact: Since  $J_w(z) = J_w(-z)$ , then

$$\int_{-\infty}^{\infty} J_w(z) dz = 2 \int_0^{\infty} J_w(z) dz,$$

so, the proof is reduced to prove that  $\int_0^{\infty} J_w(z) dz < \infty$ .

Let  $w \in R_1$  and  $x > 0$ , then

$$\begin{aligned} \int_0^x J_w(z) dz &= \int_0^\pi \int_0^x \cos(z\pi_\theta(w)) dz d\theta = \int_0^\pi \frac{\sin(x\pi_\theta(w))}{\pi_\theta(w)} d\theta \\ &= \int_0^{\theta_w^\perp} \frac{\sin(x\pi_\theta(w))}{\pi_\theta(w)} d\theta + \int_{\theta_w^\perp}^\pi \frac{\sin(x\pi_\theta(w))}{\pi_\theta(w)} d\theta := I_w^1(x) + I_w^2(x). \end{aligned}$$

The next step is to estimate  $I_w^1(x)$  and  $I_w^2(x)$ .

$$I_w^1(x) = \int_0^{a_w^\perp} \frac{\sin(x\pi_\theta(w))}{\pi_\theta(w)} d\theta + \int_{a_w^\perp}^{\theta_w^\perp} \frac{\sin(x\pi_\theta(w))}{\pi_\theta(w)} d\theta, \quad (9)$$

where  $a_w^\perp = \theta_w^\perp - \epsilon$ .

Now, by Lemma 4.1 we have that for  $\theta \in [0, a_w^\perp]$  and  $\|w\| \leq 1$ , then  $\pi_\theta(w) \geq C_1 \|w\|$  and for  $\|w\| \geq 1$ ,  $\pi_\theta(w) \geq C_1$ .

Since,  $\sin(x\pi_\theta(w)) \leq 1$ , then the first integral on the right side (9) is bounded in  $x$ . In fact:

$$\int_0^{a_w^\perp} \frac{\sin(x\pi_\theta(w))}{\pi_\theta(w)} d\theta \leq \begin{cases} \frac{\pi}{C_1 \|w\|} & \text{if } 0 < \|w\| \leq 1; \\ \frac{\pi}{C_1} & \text{if } \|w\| > 1. \end{cases} \quad (10)$$

Now we estimate the second integral on the right side of (9).

Put  $f_w(\theta) = \pi_\theta(w)$ , then  $f_w(\theta_w^\perp) = 0$  and  $f_w(\theta) > 0$  for  $\theta < \theta_w^\perp$ . Moreover, recall that by Lemma 2,  $\frac{\partial \pi}{\partial \theta}(\theta_w^\perp, w) = -\|w\| \neq 0$ , then  $f'_w(\theta_w^\perp) \neq 0$ , and put  $s = f_w(\theta)$ . Thus,

$$\int_{a_w^\perp}^{\theta_w^\perp} \frac{\sin(x\pi_\theta(w))}{\pi_\theta(w)} d\theta = - \int_0^{f_w(a_w^\perp)} \frac{\sin(xs)}{sf'_w(f_w^{-1}(s))} ds = - \int_0^{f_w(a_w^\perp)} \frac{\sin(xs)}{sg_w(s)} ds, \quad (11)$$

where  $g_w(s) = f'_w(f_w^{-1}(s))$  is  $C^\infty$ .

Now by definition of  $s$ , if  $s \in [0, f_w(a_w^\perp)]$ , then  $f_w^{-1}(s) \in [a_w^\perp, \theta_w^\perp]$ . Thus by Lemma 2 we have

$$-\|w\| \leq g_w(s) \leq -\frac{1}{2}\|w\| \quad \text{for all } s \in [0, f_w(a_w^\perp)]. \quad (12)$$

For large  $x$

$$- \int_0^{\frac{2\pi}{x}} \frac{\sin(xs)}{sg_\alpha(s)} ds = - \int_0^{\frac{\pi}{x}} \frac{\sin(xs)}{sg_\alpha(s)} ds - \int_{\pi/x}^{2\pi/x} \frac{\sin(xs)}{sg_\alpha(s)} ds$$

Since  $\sin(xs) \geq 0$  in  $[0, \frac{\pi}{x}]$  then

$$- \int_0^{\frac{\pi}{x}} \frac{\sin(xs)}{sg_\alpha(s)} ds \leq \frac{2}{\|w\|} \int_0^{\frac{\pi}{x}} \frac{\sin(xs)}{s} ds = \frac{2}{\|w\|} \int_0^\pi \frac{\sin y}{y} dy. \quad (13)$$

As well  $-\sin(xs) \geq 0$  for  $s \in [\frac{\pi}{x}, \frac{2\pi}{x}]$ , then  $- \int_{\pi/x}^{2\pi/x} \frac{\sin(xs)}{sg_w(s)} ds \leq 0$ . So, by (13)

$$- \int_0^{\frac{2\pi}{x}} \frac{\sin(xs)}{sg_w(s)} ds \leq \frac{2}{\|w\|} \int_0^\pi \frac{\sin y}{y} dy. \quad (14)$$

Let  $n \in \mathbb{N}$  such that  $n \leq \frac{xf_w(a_w^\perp)}{2\pi} \leq n+1$ , then

$$\int_0^{f_w(a_w^\perp)} \frac{\sin(xs)}{sg_w(s)} ds = \int_0^{\frac{2\pi}{x}} \frac{\sin(xs)}{sg_w(s)} ds + \sum_{k=1}^{n-1} \int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \frac{\sin(xs)}{sg_w(s)} ds + \int_{\frac{2\pi n}{x}}^{f_w(a_w^\perp)} \frac{\sin(xs)}{sg_w(s)} ds.$$

If  $\frac{2\pi n}{x} \leq f_w(a_w^\perp) \leq \frac{\pi(2n+1)}{x}$ , then  $\sin(xs) \geq 0$  and by Lemma 2, we have

$$\frac{\sin(xs)}{s\|w\|} \leq -\frac{\sin(xs)}{sg_w(s)} \leq \frac{2\sin(xs)}{s\|w\|} \quad \text{and} \quad \frac{2\sin(xs)}{s\|w\|} \leq \frac{2x\sin(xs)}{\|w\|2\pi n}.$$

This implies

$$\begin{aligned} - \int_{\frac{2\pi n}{x}}^{f_w(a_w^\perp)} \frac{\sin(xs)}{sg_w(s)} ds &\leq \int_{\frac{2\pi n}{x}}^{f_w(a_w^\perp)} \frac{x\sin(xs)}{\|w\|\pi n} ds \leq \frac{x}{\|w\|\pi n} \int_{\frac{2\pi n}{x}}^{f_w(a_w^\perp)} \sin(xs) ds \\ &\leq \frac{x}{\|w\|\pi n} \left( f_w(a_w^\perp) - \frac{2\pi n}{x} \right) = \frac{2}{\|w\|} \left( \frac{xf_w(a_w^\perp)}{2\pi n} - 1 \right) \\ &\leq \frac{2}{\|w\|} \left( \frac{2\pi(n+1)}{2\pi n} - 1 \right) = \frac{2}{\|w\|} \frac{1}{n}. \end{aligned}$$

In the case that  $f_w(a_w^\perp) \geq \frac{\pi(2n+1)}{x}$ , then  $-\int_{\frac{\pi(2n+1)}{x}}^{f_w(a_w^\perp)} \frac{\sin(xs)}{sg_w(s)} ds \leq 0$ , so

$$\begin{aligned} -\int_{\frac{2\pi n}{x}}^{f_w(a_w^\perp)} \frac{\sin(xs)}{sg_w(s)} ds &\leq -\int_{\frac{\pi(2n+1)}{x}}^{\frac{\pi(2n+1)}{x}} \frac{\sin(xs)}{sg_w(s)} ds \leq \frac{x}{\|w\| 2\pi n} \left( \frac{\pi(2n+1)}{x} - \frac{2\pi n}{x} \right) \\ &= \frac{1}{\|w\|} \frac{1}{n}. \end{aligned}$$

In any case, we have

$$\int_{\frac{2\pi n}{x}}^{f_w(a_w^\perp)} \frac{\sin(xs)}{sg_w(s)} ds \leq \frac{2}{\|w\|} \frac{1}{n}. \quad (15)$$

Now we only need to estimate  $\sum_{k=1}^{n-1} \int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \frac{\sin(xs)}{sg_w(s)} ds$ .

Put  $s_0 = \frac{2\pi k}{x}$ , then

$$\int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \frac{\sin(xs)}{sg_w(s)} ds = \int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \frac{\sin(xs)}{s_0 g_w(s_0)} ds + \int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \sin(xs) \left( \frac{1}{sg_w(s)} - \frac{1}{s_0 g_w(s_0)} \right) ds.$$

The first integral on the right term of the above equality is zero.

Now we estimate the second integral on the right side of the above equation.

By Lemma 2 we have  $g_w(s)g_w(s_0) > \frac{\|w\|^2}{4}$ , also  $ss_0 \geq \left(\frac{2\pi k}{x}\right)^2$ .

Thus,  $\frac{1}{ss_0 g_w(s)g_w(s_0)} < \frac{1}{\|w\|^2} \frac{x^2}{\pi^2 k^2}$ . Moreover,

$$\begin{aligned} |s_0 g_w(s_0) - s g_w(s)| &= |(s_0 - s)g_w(s_0) + s(g_w(s_0) - g_w(s))| \\ &\leq |s_0 - s| |g_w(s_0)| + s |g_w(s_0) - g_w(s)| \\ &\leq |s - s_0| \left( |g_w(s_0)| + s \sup_{s \in [0, f_w(a_w^\perp)]} |g'_w(s)| \right) \quad \downarrow \text{ by Lemma 2} \\ &\leq \frac{2\pi}{x} \left( \|w\| + \frac{2\pi(k+1)}{x} \|w\| \right) \\ &\leq \frac{2\pi}{x} \|w\| \left( 1 + \frac{2\pi n}{x} \right) \\ &\leq \frac{2\pi}{x} \|w\| (1 + f_w(a_w^\perp)) \\ &\leq \frac{2\pi}{x} \|w\| (1 + \|w\|) \end{aligned}$$

as,  $f_w(a_w^\perp) \leq \|w\|$ . Therefore,

$$\left| \frac{1}{sg_w(s)} - \frac{1}{s_0 g_w(s_0)} \right| = \left| \frac{sg_w(s) - s_0 g_w(s_0)}{ss_0 g_w(s)g_w(s_0)} \right| \leq \frac{2(1 + \|w\|)}{\pi \|w\|} \left( \frac{x}{k^2} \right).$$

Then,

$$\left| \int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \frac{\sin(xs)}{sg_w(s)} ds \right| \leq \frac{2(1+\|w\|)}{\pi\|w\|} \left( \frac{x}{k^2} \right) \left( \frac{2\pi(k+1)}{x} - \frac{2\pi k}{x} \right) = \frac{4(1+\|w\|)}{\|w\|} \left( \frac{1}{k^2} \right).$$

Therefore,

$$\left| \sum_{k=1}^{k=n-1} \int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \frac{\sin(xs)}{sg_w(s)} ds \right| \leq \sum_{k=1}^{k=n-1} \left| \int_{\frac{2\pi k}{x}}^{\frac{2\pi(k+1)}{x}} \frac{\sin(xs)}{sg_w(s)} ds \right| \leq A(\|w\|) \sum_{k=1}^{k=n-1} \frac{1}{k^2}, \quad (16)$$

where  $A(\|w\|) = \frac{4(1+\|w\|)}{\|w\|}$ .

Since  $\sum_{k=1}^{\infty} \frac{1}{k^2} := a < \infty$  and put  $b = \int_0^\pi \frac{\sin y}{y}$ , then the equations (14), (15), and (16) imply

$$- \int_0^{f_w(a_w^\perp)} \frac{\sin(xs)}{sg_w(s)} ds \leq \frac{2}{\|w\|} b + \frac{2}{\|w\|} \frac{1}{n} + A(\|w\|)a. \quad (17)$$

Thus, by the equation (9), (10), (11), and (17) we have

$$I_w^1(x) \leq \begin{cases} \frac{\pi}{C_1\|w\|} + \frac{2}{\|w\|}b + \frac{2}{\|w\|} \frac{1}{n} + A(\|w\|)a & \text{if } 0 < \|w\| \leq 1; \\ \frac{\pi}{C_1} + \frac{2}{\|w\|}b + \frac{2}{\|w\|} \frac{1}{n} + A(\|w\|)a & \text{if } \|w\| > 1. \end{cases} \quad (18)$$

Completely analogous using  $\tilde{a}_w^\perp$  instead of  $a_w^\perp$  and taking  $n'$  such that  $-\frac{\pi(2n'+1)}{x} \leq f_w(\tilde{a}_w^\perp) \leq -\frac{2\pi n'}{x}$ , we also obtain

$$I_w^2(x) \leq \begin{cases} \frac{\pi}{C_1\|w\|} + \frac{2}{\|w\|}b + \frac{2}{\|w\|} \frac{1}{n'} + A(\|w\|)a & \text{if } 0 < \|w\| \leq 1; \\ \frac{\pi}{C_1} + \frac{2}{\|w\|}b + \frac{2}{\|w\|} \frac{1}{n'} + A(\|w\|)a & \text{if } \|w\| > 1. \end{cases} \quad (19)$$

Since  $n, n' \rightarrow \infty$  as  $x \rightarrow \infty$ , then (18) and (19) implies

$$\int_0^\infty J_w(z) dz \leq \begin{cases} \frac{2\pi}{C_1\|w\|} + \frac{4}{\|w\|}b + 2A(\|w\|)a & \text{if } 0 < \|w\| \leq 1; \\ \frac{2\pi}{C_1} + \frac{2}{\|w\|}b + 2A(\|w\|)a & \text{if } \|w\| > 1. \end{cases} \quad (20)$$

Thus, we conclude the proof of Proposition 2.  $\square$

Put  $j_1 = 0$ ,  $j_2 = \frac{3\pi}{4}$  and  $j_3 = \frac{5\pi}{4}$ , then it is also easy to see that for  $w \in R_i$ ,

$$\int_0^\infty J_w^{j_i}(z) dz \leq \begin{cases} \frac{2\pi}{C_i\|w\|} + \frac{4}{\|w\|}b + 2A(\|w\|)a & \text{if } 0 < \|w\| \leq 1; \\ \frac{2\pi}{C_i} + \frac{2}{\|w\|}b + 2A(\|w\|)a & \text{if } \|w\| > 1, \end{cases} \quad (21)$$

$i = 1, 2, 3$ , where  $C_i$  are given in Lemma 4.

## 4 Proof of the Main Theorem.

As in the Kaufman's proof of Marstrand's theorem (cf. [Kau68]), we use the potential theory.

Put  $d = HD(K) > 1$ , assume that  $0 < M_d(K) < \infty$  and for some  $C > 0$ , we have

$$m_d(K \cap B_r(x)) \leq Cr^d$$

for  $x \in \mathbb{R}^2$  and  $0 < r \leq 1$  (cf. [Fal85]). Let  $\mu$  be the finite measure on  $\mathbb{R}^2$  defined by  $\mu(A) = m_d(K \cap A)$ ,  $A$  a measurable subset of  $\mathbb{R}^2$ . For  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ , let us denote by  $\mu_\theta$  the (unique) measure on  $\mathbb{R}$  such that  $\int f d\mu_\theta = \int (f \circ \pi_\theta) d\mu$  for every continuous function  $f$ . The theorem will follow, if we show that the support of  $\mu_\theta$  has positive Lebesgue measure for almost all  $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ , since this support is clearly contained in  $\pi_\theta(K)$ . To do this we use the following fact.

**Lemma 5.** (cf. [PT93, pg. 65]) *Let  $\eta$  be a finite measure with compact support on  $\mathbb{R}$  and*

$$\hat{\eta}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixp} d\eta(x),$$

*for  $p \in \mathbb{R}$  ( $\hat{\eta}$  is the fourier transform of  $\eta$ ). If  $0 < \int_{-\infty}^{\infty} |\hat{\eta}(p)|^2 dp < \infty$  then the support of  $\eta$  has positive Lebesgue measure.*

### Proof of the Main Theorem.

We now show that, for almost any  $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ ,  $\hat{\mu}_\theta$  is square-integrable. From the definitions we have

$$|\hat{\mu}_\theta(p)|^2 = \frac{1}{2\pi} \int \int e^{i(y-x)p} d\mu_\theta(x) d\mu_\theta(y) = \frac{1}{2\pi} \int \int e^{ip(\pi_\theta(v) - \pi_\theta(u))} d\mu(u) d\mu(v)$$

as  $\pi_{\theta+\pi}(u) = -\pi_\theta(u)$ , then

$$|\hat{\mu}_\theta(p)|^2 + |\hat{\mu}_{\theta+\pi}(p)|^2 = \frac{1}{\pi} \int \int \cos(p(\pi_\theta(v) - \pi_\theta(u))) d\mu(u) d\mu(v).$$

And so

$$\begin{aligned} \int_0^{2\pi} |\hat{\mu}_\theta(p)|^2 d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \int \int \cos(p(\pi_\theta(v) - \pi_\theta(u))) d\mu(u) d\mu(v) d\theta \\ &= \frac{1}{2\pi} \int \int \left( \int_0^{2\pi} \cos(p(\pi_\theta(v) - \pi_\theta(u))) d\theta \right) d\mu(u) d\mu(v). \end{aligned}$$

Observe now that for all  $x > 0$  and for all  $u, v$  there are  $L \in \mathbb{N}$  and  $w(u, v)$  such that

$$\int_0^x \int_0^{2\pi} \cos(p(\pi_\theta(u) - \pi_\theta(v))) d\theta dp \leq L \left| \int_0^x \int_0^{2\pi} \cos(p\pi_\theta(w(u, v))) d\theta dp \right|$$

$w(u, v)$  can be taken such that  $d(p, w) = d(u, v)$ . So, we have for  $x > 0$

$$\int_{-x}^x \int_0^{2\pi} |\hat{\mu}_\theta(p)|^2 d\theta dp \leq \frac{2L}{2\pi} \int \int \left| \int_0^x \tilde{J}_{w(u, v)}(p) dp \right| d\mu(u) d\mu(v).$$

Follows

$$\begin{aligned}
& \frac{\pi}{L} \int_{-\infty}^{\infty} \int_0^{2\pi} |\hat{\mu}_\theta(p)|^2 d\theta dp \leq \int \int \left| \int_0^{\infty} \tilde{J}_{w(u,v)}(p) dp \right| d\mu(u) d\mu(v) = \\
& = \int \int_{\{\|w\|>1\}} \left| \int_0^{\infty} \tilde{J}_{w(u,v)}(p) dp \right| d\mu(u) d\mu(v) + \int \int_{\{\|w\|\leq 1\}} \left| \int_0^{\infty} \tilde{J}_{w(u,v)}(p) dp \right| d\mu(u) d\mu(v) \\
& =: I + II.
\end{aligned} \tag{22}$$

By (8) and Remark 1

$$\begin{aligned}
I &= \int \int_{\{\|w\|>1\}} \left| \int_0^{\infty} \tilde{J}_w(p) dp \right| d\mu(u) d\mu(v) = \sum_{i=1}^3 \int \int_{\{\|w\|>1\} \cap R_i} \left| \int_0^{\infty} \tilde{J}_w(p) dp \right| d\mu(u) d\mu(v) \\
&= 2 \sum_{i=1}^3 \int \int_{\{\|w\|>1\} \cap R_i} \left| \int_0^{\infty} J_w^{j_i}(p) dp \right| d\mu(u) d\mu(v).
\end{aligned}$$

Now by (20) and (21), we have

$$I \leq 2 \sum_{i=1}^3 \int \int_{\{\|w\|>1\} \cap R_i} \left( \frac{2\pi}{C_i} + \frac{2}{\|w\|} b + 2A(\|w\|)a \right) d\mu(u) d\mu(v).$$

If  $\|w\| > 1$ , then  $\frac{1}{\|w\|} < 1$  and  $A(\|w\|) = \frac{4(1+\|w\|)}{\|w\|} < 8$ , moreover, as the support of the measure  $\mu \times \mu$  is contained in  $K \times K$  which is compact, then

$$I \leq 6 \left( 2\pi \max \left\{ \frac{1}{C_i} \right\} + 2b + 16a \right) \mu(K)^2. \tag{23}$$

We now estimate  $II$ , in fact: By (8) and Remark 1,

$$\begin{aligned}
II &= \int \int_{\{\|w\|\leq 1\}} \left| \int_0^{\infty} \tilde{J}_w(p) dp \right| d\mu(u) d\mu(v) = \sum_{i=1}^3 \int \int_{\{\|w\|\leq 1\} \cap R_i} \left| \int_0^{\infty} \tilde{J}_w(p) dp \right| d\mu(u) d\mu(v) \\
&= 2 \sum_{i=1}^3 \int \int_{\{\|w\|\leq 1\} \cap R_i} \left| \int_0^{\infty} J_w^{j_i}(p) dp \right| d\mu(u) d\mu(v).
\end{aligned}$$

Now by (20) and (21), we have

$$\begin{aligned}
II &\leq 2 \sum_{i=1}^3 \int \int_{\{\|w\|\leq 1\} \cap R_i} \left( \frac{2\pi}{C_i \|w\|} + \frac{4}{\|w\|} b + 2A(\|w\|)a \right) d\mu(u) d\mu(v) \\
&\leq 6 \int \int_{\{\|w\|\leq 1\}} \left( \left( \max \left\{ \frac{2\pi}{C_i} \right\} + 4b + 8a \right) \frac{1}{\|w\|} + 8a \right) d\mu(u) d\mu(v).
\end{aligned} \tag{24}$$

Remember that  $\|w(u, v)\| = d(u, v)$ , then

$$\int \int_{\{\|w\|\leq 1\}} \frac{1}{\|w\|} d\mu(u) d\mu(v) = \int \int_{\{d(u, v) \leq 1\}} \frac{1}{d(u, v)} d\mu(u) d\mu(v).$$

Now, for some  $0 < \beta < 1$

$$\begin{aligned}
\int_{\{\|w\| \leq 1\}} \frac{1}{d(u, v)} d\mu(v) &= \sum_{n=1}^{\infty} \int_{\beta^n \leq d(u, v) \leq \beta^{n-1}} \frac{d\mu(v)}{d(u, v)} \leq \sum_{n=1}^{\infty} \beta^{-n} \mu(B_{\beta^{n-1}}(u)) \\
&\leq C \sum_{n=1}^{\infty} \beta^{-n} (\beta^{n-1})^d \\
&\leq C \sum_{n=1}^{\infty} \beta^{-d} (\beta^{d-1})^n \text{ with } d > 1 \\
&= C \beta^{-d} \left( \frac{1}{1 - \beta^{d-1}} - 1 \right) = \frac{C}{\beta - \beta^d}.
\end{aligned}$$

Therefore,

$$\int \int_{\{\|w\| \leq 1\}} \frac{1}{\|w\|} d\mu(u) d\mu(v) \leq \mu(\mathbb{R}^2) \frac{C}{\beta - \beta^d}.$$

Also,  $\int \int_{\{\|w\| \leq 1\}} 48 d\mu(u) d\mu(v) \leq 8a\mu(K)^2 < \infty.$

Using these last two inequalities and the equation (24) we have that

$$II \leq 6 \left( \left( \max \left\{ \frac{2\pi}{C_i} \right\} + 4b + 8a \right) \frac{C}{\beta - \beta^d} + 8a\mu(K)^2 \right). \quad (25)$$

Using Fubini, the by equations (22), (23) and (25) we have

$$\begin{aligned}
\frac{\pi}{L} \int_0^{2\pi} \int_{-\infty}^{\infty} |\hat{\mu}_\theta(p)|^2 dp d\theta &\leq I + II \leq 6 \left( 2\pi \max \left\{ \frac{1}{C'_i} \right\} + 2b + 16a \right) \mu(K)^2 + \\
&6 \left( \left( \max \left\{ \frac{2\pi}{C_i} \right\} + 4b + 8a \right) \frac{C}{\beta - \beta^d} + 8a\mu(K)^2 \right) < \infty.
\end{aligned}$$

Therefore,  $\int_{-\infty}^{\infty} |\hat{\mu}_\theta(p)|^2 dp < \infty$  for almost all  $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ .

If exists  $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$  such that  $\int_{-\infty}^{\infty} |\hat{\mu}_\theta(p)|^2 dp = 0$ , then  $\int_{-\infty}^{\infty} |\varphi(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{\mu}_\theta(p)|^2 dp =$

0 where  $\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixp} \hat{\mu}_\theta(p) dp$ . This implies that  $\varphi \equiv 0$  almost every where, but  $d\mu_\theta = \varphi dx$ . This is  $\mu_\theta(\mathbb{R}) = \int_{-\infty}^{\infty} \varphi(x) dx = 0$  and this implies that  $\mu(\mathbb{R}^2) = 0$ , this contradicts the fact that  $d$ -measure of Hausdorff of  $K$  is positive.

The result follows of Lemma 5, in the case  $0 < m_d(K) < \infty$ .

In the general case, we take  $0 < m_{d'}(K') < \infty$  with  $1 < d' < d$  and  $K' \subset K$  (cf. [Fal85]). Then, by the same argument  $\pi_\theta(K')$  has positive measure for almost all  $\theta$ , and since  $\pi_\theta(K') \subset \pi_\theta(K)$ , then the same is true for  $\pi_\theta(K)$ .  $\square$



## Acknowledgments

The author is thankful to IMPA for the excellent ambient during the preparation of this manuscript. The author is also grateful to Carlos Gustavo Moreira for carefully reading the preliminary version of this work and their comments in this work. This work was financially supported by CNPq-Brazil, Capes, and the Palis Balzan Prize.

## References

- [BH99] Martin R. Bridson and André Haefliger. *Metric space of non positive curvature*. Springer vol 319, 1999.
- [Fal85] K.J. Falconer. *The geometry of fractal sets*. Cambridge University Press, 1985.
- [Kau68] R. Kaufman. On the hausdorff dimension of prejections. *Mathematika*, 15:153–155, 1968.
- [LM11] Y. Lima and C. Moreira. Yet another proof of marstrand’s theorem. *Bull Braz Math Soc*, New Series(42(2)):331–345, 2011. Sociedade Brasileira de Matemática.
- [Mar54] J.M. Marstrand. Some fundamental properties of plane sets of fractional dimension. *Proc. London Math. Soc.*, 4:257–302, 1954.
- [PadC08] Manfredo Perdigão do Carmo. *Geometria Riemannian*. Projecto Euclides, 4a edition, 2008.
- [PT93] J. Palis and F. Takens. *Hyperbolicity & sensitive chaotic dynamiscs at homo-clinic bifurcations*. Cambridge studies in advanced mathematics, 35, 1993.

**Sergio Augusto Romãña Ibarra**

Universidade Federal do Rio de Janeiro, Campus Macaé  
Av. do Aloázio, 50 - Glória, Macaé  
27930-560 Rio de Janeiro-Brasil  
E-mail: sergiori@macae.ufrj.br